

GEOMETRY OF THE SYMMETRIZED POLYDISC

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ABSTRACT. We describe all proper holomorphic mappings of the symmetrized polydisc and study its geometric properties. We also apply the obtained results to the study of the spectral unit ball in $\mathcal{M}_n(\mathbb{C}^n)$.

1. INTRODUCTION

Let \mathbb{D} be the unit disc in the complex plane \mathbb{C} . Let $\pi_n = (\pi_{n,1}, \dots, \pi_{n,n}) : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $n \geq 1$, be defined as follows

$$\pi_{n,k}(\lambda_1, \dots, \lambda_n) = \sum_{1 \leq j_1 < \dots < j_k \leq n} \lambda_{j_1} \dots \lambda_{j_k}, \quad k = 1, \dots, n.$$

Observe that π_n is a proper holomorphic mapping with the multiplicity equal to $n!$ (see e.g. [18]). Moreover, $\pi_n^{-1}(\pi_n(\mathbb{D}^n)) = \mathbb{D}^n$. Hence, $\pi_n|_{\mathbb{D}^n} : \mathbb{D}^n \rightarrow \pi_n(\mathbb{D}^n)$ is a proper holomorphic mapping. Put $\mathbb{G}_n = \pi_n(\mathbb{D}^n)$ and $\delta_n = \pi_n((\partial\mathbb{D})^n)$. The domain \mathbb{G}_n is called the *symmetrized n -disc*.

Below we present a number of results on the geometry of the symmetrized polydisc \mathbb{G}_n .

One of the main results of the paper is to give the following characterization of proper holomorphic self-mappings of the symmetrized polydisc.

Theorem 1. *Let $f : \mathbb{G}_n \rightarrow \mathbb{G}_n$ be a holomorphic mapping. Then f is proper if and only if there exists a finite Blaschke product B such that*

$$f(\pi_n(\lambda_1, \dots, \lambda_n)) = \pi_n(B(\lambda_1), \dots, B(\lambda_n)), \quad \lambda_1, \dots, \lambda_n \in \mathbb{D}.$$

In particular, f is an automorphism if and only if

$$f(\pi_n(\lambda_1, \dots, \lambda_n)) = \pi_n(h(\lambda_1), \dots, h(\lambda_n)), \quad \lambda_1, \dots, \lambda_n \in \mathbb{D},$$

where h is an automorphism of the unit disc \mathbb{D} .

Note that, if $\psi : \mathbb{D} \rightarrow \mathbb{D}$ is a holomorphic function, then the mapping $f_\psi : \mathbb{G}_n \rightarrow \mathbb{G}_n$, defined as

$$f_\psi(\pi_n(\lambda_1, \dots, \lambda_n)) = \pi_n(\psi(\lambda_1), \dots, \psi(\lambda_n)), \quad \lambda_1, \dots, \lambda_n \in \mathbb{D},$$

is a well-defined holomorphic mapping. Moreover, f_ψ is proper (resp. an automorphism) if and only if ψ is proper (resp. an automorphism).

We get Theorem 1 as a corollary of the following

Theorem 2. *Let $f : \mathbb{D}^n \rightarrow \mathbb{G}_n$ be a holomorphic mapping. Then f is proper if and only if there exist finite Blaschke products B_1, \dots, B_n such that*

$$f(\lambda_1, \dots, \lambda_n) = \pi_n(B_1(\lambda_1), \dots, B_n(\lambda_n)), \quad \lambda_1, \dots, \lambda_n \in \mathbb{D}.$$

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The symmetrized polydisc has been recently studied by many authors, especially in two-dimensional case (see e.g. [2, 3, 4], [7], [16]). Description of automorphisms in \mathbb{G}_2 was given in [12], description of proper mappings in \mathbb{G}_2 was given in [9]. We follow the ideas of the latter paper.

2. PROOFS

For the set $K = \{j_1, \dots, j_k\} \subset I_n := \{1, \dots, n\}$, $1 \leq j_1 < \dots < j_k \leq n$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ define $\lambda_K := (\lambda_{j_1}, \dots, \lambda_{j_k})$.

Let $\lambda_0 \in \mathbb{C}$, $0 \leq k \leq n$. We define $D_k^n(\lambda_0)$ to be the domain in \mathbb{C}^n of all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that there is a (uniquely defined) set $K = K(\lambda_0) \subset I_n$ with $\#K = k$ and $|\lambda_j - \lambda_0| < |\lambda_k - \lambda_0|$, $j \in K$, $k \in I_n \setminus K$.

Note that $D_n^n(\lambda_0) = D_0^n(\lambda_0) = \mathbb{C}^n$ for any $\lambda_0 \in \mathbb{C}$.

Remark 3. Let us fix $\lambda_0 \in \mathbb{C}$, $0 \leq k \leq n$. It is immediate to see that the mappings

$$(1) \quad \rho_{k,n} : D_k^n(\lambda_0) \ni \lambda \mapsto \pi_k(\lambda_K) \in \mathbb{C}^k,$$

$$\tilde{\rho}_{k,n} : D_k^n(\lambda_0) \ni \lambda \mapsto \pi_{n-k}(\lambda_{I_n \setminus K}) \in \mathbb{C}^{n-k}$$

are holomorphic. Moreover, for any mapping $\varphi \in \mathcal{O}(D, \pi_n(D_k^n(\lambda_0)))$, where D is a domain in \mathbb{C}^m , the mappings $\rho_{k,n} \circ \pi_n^{-1} \circ \varphi$ and $\tilde{\rho}_{k,n} \circ \pi_n^{-1} \circ \varphi$ are holomorphic, too.

Lemma 4. *Let $\varphi \in \mathcal{O}(D, \mathbb{C}^n)$, where D is a domain in \mathbb{C}^m . Assume that $\varphi(D) \subset \delta_n$. Then φ is constant.*

Proof. We use induction on n . The case $n = 1$ is trivial. So assume that $n \geq 2$ and that lemma is valid for dimensions $1, 2, \dots, n-1$. Let $f = (f_1, \dots, f_n)$ denote the multi-valued mapping $\pi_n^{-1} \circ \varphi$. Let k denote the maximal number of elements of a set $K \subset I_n$ such that for some $\mu_0 \in \mathbb{D}$ all the coordinates $f_j(\mu_0)$, $j \in K$, are equal. Without loss of generality we may assume that $f_1(\mu_0) = \dots = f_k(\mu_0)$ and $f_l(\mu_0) \neq f_1(\mu_0)$, $l \geq k+1$. Shrinking D , if necessary, we may assume that $\varphi(D) \subset \pi_n(D_k^n(f_1(\mu_0)))$, so in view of Remark 3 (with $\lambda_0 = f_1(\mu_0)$) we get that $\rho_{k,n} \circ f$ is holomorphic. In particular, $h := (\rho_{k,n} \circ f)_1$ is holomorphic on D and $|h|$ attains its maximum (equal to k) at 0. Therefore, losing no generality, we may assume that f_1, \dots, f_k are constant on D . In the case $k = n$ this finishes the proof. So assume that $1 \leq k < n$. Note that the function $\tilde{\varphi} := \tilde{\rho}_{k,n} \circ f$ is holomorphic on D and $\tilde{\varphi}(D) \subset \delta_{n-k}$. Then the inductive assumption implies that $\tilde{\varphi}$ is constant, which easily implies that f_{k+1}, \dots, f_n are constant, which finishes the proof. \square

Let us define one more mapping. For $0 \leq k \leq n$ and for $w = \pi_k(\lambda)$, $z = \pi_{n-k}(\mu)$ define

$$(2) \quad p_{k,n} : \mathbb{C}^k \times \mathbb{C}^{n-k} \ni (w, z) \mapsto \pi_n(\lambda, \mu) \in \mathbb{C}^n.$$

Note that $p_{k,n}$ is a holomorphic mapping, $p_{k,n}(\mathbb{G}_k \times \mathbb{G}_{n-k}) = \mathbb{G}_n$, $p_{k,n}(\bar{\mathbb{G}}_k \times \bar{\mathbb{G}}_{n-k}) = \bar{\mathbb{G}}_n$.

Lemma 5. *Let $\varphi \in \mathcal{O}(D, \mathbb{C}^n)$, where D is a domain in \mathbb{C}^m , be such that $\varphi(D) \subset \partial \mathbb{G}_n$. Then there are a k with $1 \leq k \leq n$, $\theta \in \delta_k$, and $\psi \in \mathcal{O}(D, \mathbb{G}_{n-k})$ such that $\varphi = p_{k,n} \circ (\theta, \psi)$.*

Proof. The case $n = 1$ is trivial. So assume that $n \geq 2$. Define as in Lemma 4 $(f_1, \dots, f_n) := \pi_n^{-1} \circ \varphi$. Put $N(\lambda) := \#\{j \in I_n : |f_j(\lambda)| = 1\}$, $\lambda \in D$. Define $k := \max\{N(\lambda) : \lambda \in D\}$. Let $k = N(\mu_0)$, $\mu_0 \in D$. Obviously, $1 \leq k \leq n$.

Denote $u(\lambda) := \max_{1 \leq j_1 < \dots < j_k \leq n} \{|f_{j_1}(\lambda)| \cdot \dots \cdot |f_{j_k}(\lambda)|\}$, $\lambda \in D$. Then u is plurisubharmonic in D , $u \leq 1$ on D and $u(\mu_0) = 1$. Then $u \equiv 1$ on D . Therefore, for any $\lambda \in D$ there is a set $K \subset I_n$ with k elements such that $|f_j(\lambda)| = 1$, $j \in K$ and $|f_j(\lambda)| < 1$, $j \in I_n \setminus K$. Then Lemma 4 finishes the proof in the case $k = n$ (because $\varphi(D) \subset \delta_n$). So assume that $k < n$. Applying Remark 3 (for $\lambda_0 = 0$) we see that $\varphi = p_{k,n} \circ (\eta, \psi)$, where $\eta = \rho_{k,n} \circ f$, $\psi = \tilde{\rho}_{k,n} \circ f$ are holomorphic on D . It easily follows from the definition that $\psi(D) \subset \mathbb{G}_{n-k}$. Moreover, $\eta(D) \subset \delta_k$, so, in view of Lemma 4, η is constant. \square

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. Note that $\lambda_1, \dots, \lambda_n$ are roots of the polynomial equation

$$(3) \quad z^n - (\pi_n(\lambda))_1 z^{n-1} + (\pi_n(\lambda))_2 z^{n-2} + \dots + (-1)^{n-1} (\pi_n(\lambda))_{n-1} z + (-1)^n (\pi_n(\lambda))_n = 0, \quad z \in \mathbb{C}.$$

Therefore, Lemma 5 easily implies the following

Lemma 6. *Let $\varphi \in \mathcal{O}(D, \mathbb{C}^n)$, where D is a domain in \mathbb{C}^m , be such that $\varphi(D) \subset \partial \mathbb{G}_n$. Then there is a constant $C \in \partial \mathbb{D}$ such that*

$$(4) \quad C^n - \varphi_1(\lambda) C^{n-1} + \varphi_2(\lambda) C^{n-2} + \dots + (-1)^{n-1} \varphi_{n-1}(\lambda) C + (-1)^n \varphi_n(\lambda) = 0, \quad \lambda \in D.$$

The proof of Theorem 2 is based on the following result.

Proposition 7. *Let $f : \mathbb{D}^n \rightarrow \mathbb{G}_n$ be a proper holomorphic mapping. Then there exists a bounded holomorphic function B on \mathbb{D} such that*

$$(5) \quad B^n(z_n) - B^{n-1}(z_n) f_1(z) + \dots + (-1)^{n-1} B(z_n) f_{n-1}(z) + (-1)^n f_n(z) = 0 \\ z = (z_1, \dots, z_n) \in \mathbb{D}^n.$$

Moreover, B is non-constant.

Proof. We use methods similar to the ones of Remmert-Stein (see e.g. [15]). Take a sequence $\mathbb{D} \ni z_n^\nu \rightarrow \partial \mathbb{D}$. Then there exists a subsequence ν_k such that $f(\cdot, z_n^{\nu_k}) \rightarrow \varphi$ locally uniformly on \mathbb{D}^{n-1} . Note that $\varphi : \mathbb{D}^{n-1} \rightarrow \partial \mathbb{G}_n$ is a holomorphic mapping. Hence, in view of Lemma 6 there exists a constant $C \in \partial \mathbb{D}$ such that

$$(6) \quad C^n - C^{n-1} \varphi_1 + \dots + (-1)^{n-1} C \varphi_{n-1} + (-1)^n \varphi_n \equiv 0 \quad \text{on } \mathbb{D}^{n-1}.$$

Note that

$$(7) \quad -C^{n-1} \frac{\partial \varphi_1}{\partial z_j} + \dots + (-1)^{n-1} C \frac{\partial \varphi_{n-1}}{\partial z_j} + (-1)^n \frac{\partial \varphi_n}{\partial z_j} \equiv 0 \quad \text{on } \mathbb{D}^{n-1} \\ j = 1, 2, \dots, n-1.$$

Since f is proper, we have

$$\det \left[\frac{\partial f_i}{\partial z_j} \right]_{i,j=1,\dots,n} \not\equiv 0 \quad \text{on } \mathbb{D}^n.$$

So, there exists a $k \in \{1, \dots, n\}$ such that

$$\det \left[\frac{\partial f_i}{\partial z_j} \right]_{i=1, \dots, n, j=1, \dots, n-1, i \neq k} \neq 0 \quad \text{on } \mathbb{D}^n.$$

We want to solve the equations (w.r.t. C)

$$(8) \quad \begin{cases} \sum_{m=1, m \neq k}^n (-1)^m C^{n-m} \frac{\partial \varphi_m}{\partial z_1} = (-1)^{k+1} C^{n-k} \frac{\partial \varphi_k}{\partial z_1} \\ \dots\dots\dots \\ \sum_{m=1, m \neq k}^n (-1)^m C^{n-m} \frac{\partial \varphi_m}{\partial z_{n-1}} = (-1)^{k+1} C^{n-k} \frac{\partial \varphi_k}{\partial z_{n-1}} \end{cases}.$$

Let

$$\Phi_m(\varphi_1, \dots, \varphi_n) := \det \left[\frac{\partial \varphi_i}{\partial z_j} \right]_{i=1, \dots, n, i \neq m, j=1, \dots, n-1} \quad \text{on } \mathbb{D}^{n-1}.$$

Then $(-1)^m C^{n-m} \Phi_k(\varphi_1, \dots, \varphi_n) = (-1)^k C^{n-k} \Phi_m(\varphi_1, \dots, \varphi_n)$.

If $k > 1$ then take $m = k - 1$. Hence,

$$(9) \quad \Phi_{k-1}(\varphi_1, \dots, \varphi_n) = -C \Phi_k(\varphi_1, \dots, \varphi_n) \quad \text{on } \mathbb{D}^{n-1}.$$

We have

$$(10) \quad \Phi_{k-1}(\varphi_1, \dots, \varphi_n) \frac{\partial \Phi_k(\varphi_1, \dots, \varphi_n)}{\partial z_j} = \Phi_k(\varphi_1, \dots, \varphi_n) \frac{\partial \Phi_{k-1}(\varphi_1, \dots, \varphi_n)}{\partial z_j} \quad \text{on } \mathbb{D}^{n-1}$$

for any $j = 1, \dots, n-1$. Note that the equations in (10) hold for any choice of possible sequences $z_n^\nu \rightarrow \partial \mathbb{D}$. Therefore, the maximum principle for holomorphic functions implies that similar equations hold for f_1, \dots, f_n .

So,

$$(11) \quad \Phi_{k-1}(f_1, \dots, f_n) \frac{\partial \Phi_k(f_1, \dots, f_n)}{\partial z_j} = \Phi_k(f_1, \dots, f_n) \frac{\partial \Phi_{k-1}(f_1, \dots, f_n)}{\partial z_j} \quad \text{on } \mathbb{D}^n$$

for any $j = 1, \dots, n-1$. Put $A = \{\Phi_k(f_1, \dots, f_n) = 0\}$ and

$$(12) \quad B = -\frac{\Phi_{k-1}(f_1, \dots, f_n)}{\Phi_k(f_1, \dots, f_n)}.$$

Note that A is a proper analytic subset of \mathbb{D}^n and that B is a holomorphic function on $\mathbb{D}^n \setminus A$. Moreover, $\frac{\partial B}{\partial z_j} = 0$ on $\mathbb{D}^n \setminus A$ for any $j = 1, \dots, n-1$. Hence, B depends only on z_n .

Moreover, (use (6), (9))

$$\Phi_{k-1}^n + \varphi_1 \Phi_{k-1}^{n-1} \Phi_k + \dots + \varphi_{n-1} \Phi_{k-1} \Phi_k^{n-1} + \varphi_n \Phi_k^n \equiv 0 \quad \text{on } \mathbb{D}^{n-1}.$$

Hence, similar result holds for f_1, \dots, f_n . From this we get

$$(13) \quad B^n(z_n) - B^{n-1}(z_n) f_1 + \dots + (-1)^{n-1} B(z_n) f_{n-1} + (-1)^n f_n \equiv 0 \quad \text{on } \mathbb{D}^n \setminus A.$$

Note that B is a bounded function on $\mathbb{D}^n \setminus A$, so it extends holomorphically to \mathbb{D}^n .

If $k = 1$ then take $m = 2$. Later we prove in a similar way. \square

Proof of Theorem 1. From Proposition 7 we get that there are holomorphic functions B_1, \dots, B_n defined on \mathbb{D} such that

$$(14) \quad \begin{aligned} B_m^n(z_m) - B_m^{n-1}(z_m)f_1(z) + \dots + (-1)^{n-1}B_m(z_m)f_{n-1}(z) + (-1)^n f_n(z) &\equiv 0 \\ z = (z_1, \dots, z_m) &\in \mathbb{D}^n, \quad m = 1, \dots, n. \end{aligned}$$

Hence, $f = \pi(B_1, \dots, B_n)$. So, $B = (B_1, \dots, B_n) : \mathbb{D}^n \rightarrow \mathbb{D}^n$ is a proper holomorphic mapping. From this we have the proof. \square

3. THE SHILOV BOUNDARY

We start with the description of the Shilov boundary of \mathbb{G}_n . The description of the Shilov boundary in the special case $n = 2$ can be found in [13] (see also [3]).

Lemma 8. *The set δ_n is the Shilov boundary of \mathbb{G}_n .*

Proof. It is easy to see that the modulus of any function from $\mathcal{C}(\bar{\mathbb{G}}_n) \cap \mathcal{O}(\mathbb{G}_n)$ attains its maximum in δ_n . To finish the proof it is sufficient to show that for any $z^0 \in \delta_n$ there is a function $F \in \mathcal{C}(\bar{\mathbb{G}}_n) \cap \mathcal{O}(\mathbb{G}_n)$ such that $|F|$ attains its strict maximum at z^0 .

Fix $z^0 := \pi_n(\lambda_1^0, \dots, \lambda_n^0) \in \delta_n$, $|\lambda_1^0| = \dots = |\lambda_n^0| = 1$. Let B_1 be a finite Blaschke product such that $B_1(\lambda_1^0) = \dots = B_1(\lambda_n^0) = 1$ (see [1], [19]). Let $B_1^{-1}(1) = \{\lambda_1^0, \dots, \lambda_m^0\}$, where $m \geq n$. Let B_2 be a finite Blaschke product such that $B_2(\lambda_1^0) = \dots = B_2(\lambda_n^0) = 1$ and $B_2(\lambda_{n+1}^0) = \dots = B_2(\lambda_m^0) = -1$ (use once more [1], [19]). Let $B := B_1 + B_2$. Note that $|B| \leq 2$ on $\bar{\mathbb{D}}$ and $B^{-1}(2) = \{\lambda_1^0, \dots, \lambda_n^0\}$. Define

$$F(z) := 1 + \sum_{j=1}^n B(\lambda_j),$$

where $z = \pi(\lambda_1, \dots, \lambda_n) \in \bar{\mathbb{G}}_n$, $\lambda_1, \dots, \lambda_n \in \bar{\mathbb{D}}$. Then $F \in \mathcal{C}(\bar{\mathbb{G}}_n) \cap \mathcal{O}(\mathbb{G}_n)$ and $|F|$ attains its strict maximum (equal to $1 + 2n$) at z^0 . \square

4. THE BERGMAN KERNEL

It easily follows from the properties of proper holomorphic mappings that

$$(15) \quad \begin{aligned} \mathcal{J}_n &:= \{\lambda \in \mathbb{C}^n : \det \pi'(\lambda) = 0\} = \\ &\quad \{\lambda \in \mathbb{C}^n : \lambda_j = \lambda_k \text{ for some } j \neq k, j, k = 1, \dots, n\}. \end{aligned}$$

Denote by K_D the Bergman kernel of the domain $D \subset \mathbb{C}^n$ (see e.g. [11]).

Proposition 9.

$$K_{\mathbb{G}_n}(\pi_n(\lambda), \pi_n(\mu)) = \frac{\det \left[\frac{1}{(1 - \lambda_j \bar{\mu}_k)^2} \right]_{1 \leq j, k \leq n}}{\pi^n \det \pi'_n(\lambda) \det \bar{\pi}'_n(\mu)}$$

for any $\lambda, \mu \in \mathbb{D}_n \setminus \mathcal{J}_n$.

Let Σ_n denote the group of all permutations of the set I_n . For $\sigma \in \Sigma_n$, $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ denote $\lambda_\sigma := (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)})$.

Proof. From the formula for the Bergman kernel of the polydisc and from the formula for the behavior of the Bergman kernel under proper holomorphic mappings (see [6]) we get for any $\lambda, \mu \in \mathbb{D}_n \setminus \mathcal{J}_n$

$$(16) \quad K_{\mathbb{G}_n}(\pi_n(\lambda), \pi_n(\mu)) = \frac{1}{\det \pi'_n(\lambda)} \sum_{\sigma \in \Sigma_n} K_{\mathbb{D}^n}(\lambda, \mu_\sigma) \frac{1}{\det \pi'_n(\mu_\sigma)} =$$

$$\frac{1}{\pi^n \det \pi'_n(\lambda) \det \pi'_n(\mu)} \sum_{\sigma \in \Sigma_n} \frac{(-1)^{\text{sgn } \sigma}}{(1 - \lambda_j \bar{\mu}_{\sigma(k)})^2} = \frac{\det \left[\frac{1}{(1 - \lambda_j \bar{\mu}_k)^2} \right]_{1 \leq j, k \leq n}}{\pi^n \det \pi'_n(\lambda) \det \pi'_n(\mu)}.$$

□

The formula above extends analytically to a formula on $\mathbb{G}_n \times \mathbb{G}_n$. It would be interesting to find a more handy formula for $K_{\mathbb{G}_n}$. Below we deliver such a formula in the case $n = 2$. We start with the simplification of the denominator in the formula for $K_{\mathbb{G}_n}$.

Lemma 10. $\det \pi'_1(\lambda) = 1$, $\lambda \in \mathbb{C}$, and $\det \pi'_n(\lambda) = \prod_{1 \leq j < k \leq n} (\lambda_j - \lambda_k)$, $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, for any $n \geq 2$.

Proof. We prove by induction on n . For $n = 2$ we get it by easy computations. Put $\pi_{n,0} \equiv 1$ for any $n \geq 1$. Note that

$$(17) \quad \frac{\partial \pi_{n,k}}{\partial \lambda_1}(\lambda_1, \dots, \lambda_n) - \frac{\partial \pi_{n,k}}{\partial \lambda_n}(\lambda_1, \dots, \lambda_n) = (\lambda_n - \lambda_1) \frac{\partial \pi_{n-1,k-1}}{\partial \lambda_1}(\lambda_1, \dots, \lambda_{n-1}).$$

Similar equation holds for any pair (λ_j, λ_n) . From this we get

$$(18) \quad \det \pi'_n(\lambda_1, \dots, \lambda_n) = \det \left[\frac{\partial \pi_{n,k}}{\partial \lambda_j}(\lambda_1, \dots, \lambda_n) \right]_{j,k=1,\dots,n}$$

$$= (\lambda_1 - \lambda_n) \dots (\lambda_{n-1} - \lambda_n) \det \left[\frac{\partial \pi_{n-1,k}}{\partial \lambda_j}(\lambda_1, \dots, \lambda_{n-1}) \right]_{j,k=1,\dots,n-1}$$

$$= (\lambda_1 - \lambda_n) \dots (\lambda_{n-1} - \lambda_n) \det \pi'_{n-1}(\lambda_1, \dots, \lambda_{n-1}).$$

□

In the case $n = 2$ elementary calculation shows.

Proposition 11.

$$(19) \quad K_{\mathbb{G}_2}(\pi_2(\lambda), \pi_2(\mu)) = \frac{2 - (\bar{\mu}_1 + \bar{\mu}_2)(\lambda_1 + \lambda_2) + 2\lambda_1\lambda_2\bar{\mu}_1\bar{\mu}_2}{\pi^2((1 - \lambda_1\bar{\mu}_1)(1 - \lambda_2\bar{\mu}_2)(1 - \lambda_1\bar{\mu}_2)(1 - \lambda_2\bar{\mu}_2))^2},$$

$$\lambda, \mu \in \mathbb{D}_2.$$

In particular, \mathbb{G}_2 is the Lu-Qi-Keng domain, i.e. $K_{\mathbb{G}_2}(z, w) \neq 0$ for any $z, w \in \mathbb{G}_2$.

Proof. To get the desired formula it is sufficient to apply the formula from Proposition 9, Lemma 10, and then make elementary calculations. To prove that the domain \mathbb{G}_2 is Lu Qi-keng, it is sufficient, in view of the form of the automorphisms of \mathbb{G}_2 , to verify that $K_{\mathbb{G}_2}(\pi_2(\lambda), \pi_2(\mu)) \neq 0$, $\lambda, \mu \in \mathbb{D}^2$ under the additional assumption $\mu_2 = 0$. But this easily follows from the obtained formula (19). □

Remark 12. It would be interesting to find a more effective formula for the Bergman kernel for \mathbb{G}_n , $n \geq 3$. Moreover, the problem whether the domain \mathbb{G}_n , $n \geq 3$, is Lu Qi-keng, is open, too.

5. THE SPECTRAL BALL

Define $\Omega_n := \{W \in \mathcal{M}_n(\mathbb{C}) : r(W) < 1\}$, where $r(W)$ denotes the spectral radius in $\mathcal{M}_n(\mathbb{C})$.

Denote also the following mapping

$$\Psi_n : \Omega_n \ni W \mapsto \pi_n(\sigma(W)) \in \mathbb{G}_n,$$

where $\sigma(W)$ denotes the spectrum of W . Note that Ψ_n is a holomorphic mapping, which is onto but not one-to-one.

Lemma 13. *Let $W \in \Omega_n$. Put $\sigma(W) := \{\lambda_1, \dots, \lambda_n\}$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ (it may happen that $\lambda_j = \lambda_k$ for some $j \neq k$). Then there is a holomorphic mapping $f : \mathbb{C} \mapsto \mathcal{M}_n(\mathbb{C})$ such that $f(0) = W$, $f(1) = \text{diag}(\lambda_1, \dots, \lambda_n)$ and $\sigma(f(\lambda)) = \{\lambda_1, \dots, \lambda_n\}$ for any $\lambda \in \mathbb{C}$.*

Proof. We proceed as in [8]. There is an invertible matrix $X \in \mathcal{M}_n(\mathbb{C})$ and an upper triangular matrix

$$S = \begin{bmatrix} \lambda_1 & x_{1,2} & x_{1,3} & \dots & x_{1,n-1} & x_{1,n} \\ 0 & \lambda_2 & x_{2,3} & \dots & x_{2,n-1} & x_{2,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_{n-1} & x_{n-1,n} \\ 0 & 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

such that $W = XSX^{-1}$. Then we can find a matrix $Y \in \mathcal{M}_n(\mathbb{C})$ such that $X = e^Y$ (see e.g. [5]) and, therefore, $W = e^Y S e^{-Y}$. Consider the holomorphic mappings $g_{j,k} : \mathbb{C} \mapsto \mathbb{C}$, $j = 1, \dots, n-1$, $k = j+1, \dots, n$ such that $g_{j,k}(0) = 0$ and $g(1) = x_{j,k}$. Then we define $f : \mathbb{C} \mapsto \Omega_n$ as follows

$$f(\lambda) := e^{\lambda Y} \begin{bmatrix} \lambda_1 & g_{1,2}(\lambda) & g_{1,3}(\lambda) & \dots & g_{1,n-1}(\lambda) & g_{1,n}(\lambda) \\ 0 & \lambda_2 & g_{2,3}(\lambda) & \dots & g_{2,n-1}(\lambda) & g_{2,n}(\lambda) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_{n-1} & g_{n-1,n}(\lambda) \\ 0 & 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix} e^{-\lambda Y}.$$

□

Proposition 14. *Let $F : \Omega_n \mapsto \Omega_n$ be holomorphic. Then there is a holomorphic mapping $\tilde{F} : \mathbb{G}_n \mapsto \mathbb{G}_n$ such that*

$$(20) \quad \tilde{F}(\Psi_n(W)) = \Psi_n(F(W)), \quad W \in \mathcal{M}_n(\mathbb{C}).$$

Proof. It is sufficient to show that for any $W \in \Omega_n$

$$\sigma(F(W)) = \sigma(F(\text{diag}(\lambda_1, \dots, \lambda_n))),$$

where $\sigma(W) = \{\lambda_1, \dots, \lambda_n\}$. In view of the previous lemma there is a holomorphic mapping $f : \mathbb{C} \mapsto \Omega_n$ such that $f(0) = W$, $f(1) = \text{diag}(\lambda_1, \dots, \lambda_n)$.

The mapping $\Psi_n \circ F \circ f : \mathbb{C} \mapsto \mathbb{G}_n$ is holomorphic. Since \mathbb{G}_n is bounded we get that $\Psi_n \circ F \circ f$ is constant. In particular, $\pi_n(\sigma(\text{diag}(\lambda_1, \dots, \lambda_n))) = \pi_n(\sigma(F(W)))$, which finishes the proof. □

For a finite set $\emptyset \neq P \subset D$ denote $g_D(P, \cdot)$ the pluricomplex Green function with the pole set in P (see e.g. [10]). If $P = \{p\}$ then we put $g_D(p, \cdot) = g_D(\{p\}, \cdot)$.

Proposition 15. *Let $W_1, W_2 \in \Omega_n$. Then*

$$g_{\Omega_n}(W_1, W_2) = g_{\Omega_n}(W_1, \text{diag}(\lambda_1, \dots, \lambda_n)),$$

where $\sigma(W_2) = \{\lambda_1, \dots, \lambda_n\}$.

Proof. Let $f : \mathbb{C} \mapsto \Omega_n$ be such that $f(0) = W_2$, $f(1) = \text{diag}(\lambda_1, \dots, \lambda_n)$ (use Lemma 13). Since the function $u := g_{\Omega_n}(W_1, f(\cdot)) : \mathbb{C} \mapsto [-\infty, 0]$ is subharmonic, u must be constant. Therefore, $g_{\mathbb{G}_n}(W_1, W_2) = u(0) = u(1) = g_{\Omega_n}(W_1, \text{diag}(\lambda_1, \dots, \lambda_n))$. \square

A domain $D \subset \mathbb{C}^n$ is called hyperconvex if there exists a negative plurisubharmonic exhaustion u of D , i.e. $\{z \in D : u(z) < -\epsilon\} \Subset D$ for any $\epsilon > 0$ (see e.g. [14]). Note that any hyperconvex domain is pseudoconvex.

We have the following result.

Proposition 16. *\mathbb{G}_n is a hyperconvex domain for any $n \geq 1$.*

Proof. We know that $\mathbb{G}_n = \pi_n(\mathbb{D}^n)$. Let $q(z) = \max\{|z_1|, \dots, |z_n|\}$, where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Put $u(w) = \max \log q(\pi_n^{-1}(w))$, where $w \in \mathbb{C}^n$. Note that $u|_{\mathbb{G}_n}$ is a negative plurisubharmonic exhaustion of \mathbb{G}_n (use Proposition 2.9.26 in [14]). \square

The following result is a partial generalization of the main result in [17].

Theorem 17. *Let $F : \Omega_n \mapsto \Omega_n$ be a proper holomorphic mapping. Then there is a finite Blaschke product B such that $\sigma(F(W)) = B(\sigma(W))$, $W \in \Omega_n$.*

Proof. Let $\tilde{F} : \mathbb{G}_n \mapsto \mathbb{G}_n$ be as in Proposition 14. Because of Theorem 1 it is sufficient to show that \tilde{F} is proper. It is sufficient to show that for any sequence $\mathbb{G}_n \supset (z^\nu)$ such that $z^\nu \rightarrow \partial \mathbb{G}_n$ there is a subsequence (z_{ν_k}) such that $\tilde{F}(z_{\nu_k}) \rightarrow \partial \mathbb{G}_n$. Fix such a sequence (z^ν) . Let $(W^\nu) \subset \Omega_n$ be such a sequence that $\Psi_n(W^\nu) = z^\nu$. Let $Q \in \Omega_n$ be any point such that $Q \notin F(\{\det F' = 0\})$. Let N denote the multiplicity of the proper mapping F . Denote $\{P_1, \dots, P_N\} = F^{-1}(Q)$. Denote $\sigma(F(W^\nu)) := \{\lambda_1^\nu, \dots, \lambda_n^\nu\}$. Then because of Proposition 15 and the behaviour of the Green function under proper holomorphic mappings (see [10]) we get

$$(21) \quad g_{\Omega_n}(Q, \text{diag}(\lambda_1^\nu, \dots, \lambda_n^\nu)) = g_{\Omega_n}(Q, F(W^\nu)) = \\ g_{\Omega_n}(\{P_1, \dots, P_N\}, W^\nu) \geq \sum_{j=1}^N g_{\Omega_n}(P_j, W^\nu) \geq \sum_{j=1}^N g_{\mathbb{G}_n}(\Psi_n(P_j), z^\nu).$$

Now the hyperconvexity of \mathbb{G}_n and the convergence $z^\nu \rightarrow \partial \mathbb{G}_n$ imply that the last expression tends to 0 as $\nu \rightarrow \infty$ (see e.g. [14]).

Choosing a subsequence, if necessary, we may assume that $\lambda_j^\nu \rightarrow \lambda_j$, $j = 1, \dots, n$, where $|\lambda_j| \leq 1$. Now the assumption $|\lambda_j| < 1$, $j = 1, \dots, n$, would imply, because of the upper-semicontinuity of the Green function, that $g_{\Omega_n}(Q, \text{diag}(\lambda_1, \dots, \lambda_n)) = 0$ – contradiction. Consequently, $\tilde{F}(z^\nu) = \Psi_n(W^\nu) = \Psi_n(\text{diag}(\lambda_1^\nu, \dots, \lambda_n^\nu)) \rightarrow \Psi_n(\text{diag}(\lambda_1, \dots, \lambda_n)) \in \partial \mathbb{G}_n$. \square

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